

HANKEL DETERMINANTS OF THE CANTOR SEQUENCE

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ABSTRACT. In the paper, we give the recurrent equations of the Hankel determinants of the Cantor sequence, and show that the Hankel determinants as a double sequence is 3-automatic. With the help of the Hankel determinants, we prove that the irrationality exponent of the Cantor number, i.e. the transcendental number with Cantor sequence as its b -ary expansion, equals 2.

1. INTRODUCTION.

Let σ be the *Cantor substitution* defined over the alphabet $\mathcal{A} = \{a, b\}$ by

$$\sigma : a \mapsto aba, b \mapsto bbb.$$

Since the word $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$, i.e., $\sigma^{n+1}(a) = \sigma^n(a)w$ where w is a finite word over the alphabet \mathcal{A} , the sequence of words $(\sigma^n(a))_{n \in \mathbb{N}}$ converges to the infinite sequence

$$\mathbf{c} := c_0c_1c_2 \cdots \in \mathcal{A}^{\mathbb{N}},$$

called the *Cantor sequence*. In this paper, we take $a = 1$ and $b = 0$. Then the sequence $(c_n)_{n \geq 0}$ takes the value 1 if the 3-ary expansion of n contains no '1', and 0 otherwise (see [2], [6]). Here are the first few terms:

n	0	1	2	3	4	5	6	7	8	\cdots
c_n	1	0	1	0	0	0	1	0	1	\cdots

The Cantor sequence is an *automatic sequence* (see [2]), i.e., it can be generated by a finite automaton. In detail, the Cantor sequence can be recognized by the 3-automaton in Figure 1 in direct reading with the initial state a and the output map $a \mapsto 1, b \mapsto 0$. The Cantor sequence can be recognized as the discretization of

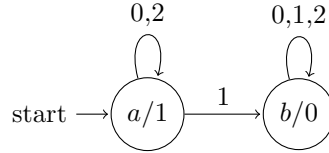


FIGURE 1. Automaton Generating the Cantor Sequence.

the Cantor ternary set. In fact, we can construct the Cantor ternary set from the

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Cantor sequence as a recurrent set, following the way introduced by F. M. Dekking in [5].

In this paper, we discuss some property of the Hankel determinants of the Cantor sequence.

1.1. Hankel determinants of Cantor sequence. For a sequence of complex numbers $\mathbf{u} = u_0 u_1 \cdots u_n \cdots$, the corresponding (p, n) -order *Hankel matrix* H_n^p is given by

$$H_n^p = \begin{pmatrix} u_p & u_{p+1} & \cdots & u_{p+n-1} \\ u_{p+1} & u_{p+2} & \cdots & u_{p+n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{p+n-1} & u_{p+n} & \cdots & u_{p+2n-2} \end{pmatrix}$$

where $n \geq 1$ and $p \geq 0$. And $|H_n^p|$ denote the determinant of the matrix H_n^p .

Positive definiteness of Hankel matrices associated with a sequence are strong connected the moment problem(see [11]). The properties of Hankel determinants are also connected to the combinatoric properties of the sequences and to the Padé approximation(see [3], [4]). In [8], Kamae, Tamura and Wen studied the properties of Hankel determinants for the Fibonacci word and give a quantitative relation between the Hankel determinant and the Padé pair. Later, Tamura [12] generalized the results for a class of special sequences. Allouche, Peyrière, Wen and Wen studied the properties of Hankel determinants $|\mathcal{E}_n^p|$ of the Thue-Morse sequence in [1]. They proved that the Hankel determinants $|\mathcal{E}_n^p|$ (modulo 2) recognized as a two-dimensional sequence (or *double sequence*) was 2-automatic. Recently, Gou, Wen and Wu [7] proved that the Hankel determinants (modulo 2) of regular paper-folding sequence are also periodic.

Let Γ_n^p be the (p, n) -order Hankel matrix of Cantor sequence. Our purpose is to discuss the property of the two-dimensional sequence $\{|\Gamma_n^p|\}_{n,p \geq 0}$. We have the following results.

Main Results.

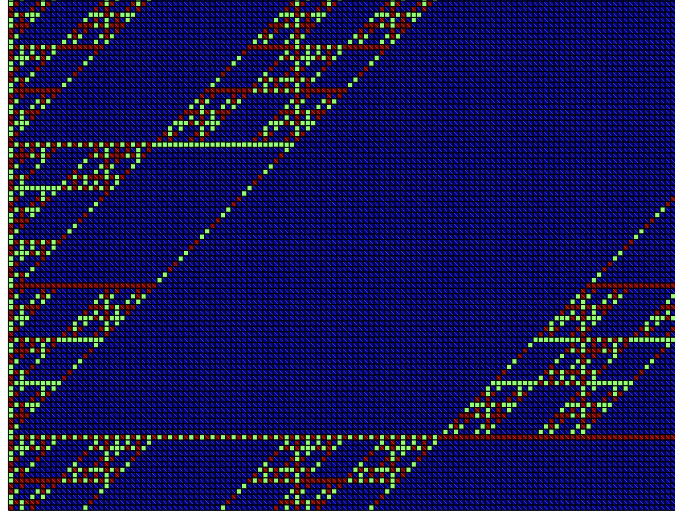
- (1) For each $p \geq 0$, the sequence $\{|\Gamma_n^p|\}_{n \geq 0}$ is periodic(see Theorem 2).
- (2) The two-dimensional sequence(modulo 3) $\{|\Gamma_n^p|\}_{n,p \geq 0}$ is 3-automatic (see Theorem 3).

Figure 2 shows $|\Gamma_n^p|$ modulo 3 for $1 \leq n \leq 96$ and $0 \leq p \leq 127$, where 0's are replaced by blue squares, 1's are replaced by green squares and 2's are replaced by red squares. Columns refer sequences $\{|\Gamma_n^p|\}_{n \geq 0}$.

This article is organized as follows. Definitions and preliminaries are given in Section 2. In Section 3, we give recurrence formulae of Hankel determinants (modulo 3) of the Cantor sequence. Section 4 is mainly devoted to characterize the periodicity and automaticity properties of Hankel determinants (modulo 3) of the Cantor sequence. In Section 5, we will prove that the Padé approximants to the generating series of the Cantor sequence exist and the irrationality exponent of the Cantor number and the difference sequence are both equal to 2.

2. PRELIMINARIES.

Assume that M is an $n \times n$ -matrix. Denote by M^t the transpose of M . Let $M^{(i)}$ be the $n \times (n-1)$ -matrix obtained by deleting the i -th column of M , and $M_{(i)}$ be

FIGURE 2. $|\Gamma_n^p|(\bmod 3)$ for $1 \leq n \leq 96$ and $0 \leq p \leq 127$.

the $(n-1) \times n$ -matrix obtained by deleting the i -th row of M . $|M|$ denote the determinant of the matrix M and $\mathbf{0}_{m \times n}$ denote the $m \times n$ zero matrix.

It is easy to check that the Cantor sequence $\mathbf{c} = c_0 c_1 \cdots c_n \cdots \in \{0, 1\}^{\mathbb{N}}$ can also be defined by the following recurrence equations:

$$(2.1) \quad c_0 = 1, \quad c_{3n} = c_n, \quad c_{3n+1} = 0, \quad c_{3n+2} = c_n, \quad \text{for } n \geq 0.$$

For further consideration, we need another sequence \mathbf{d} defined by $d_n = c_n + c_{n+2}$, and simultaneously we have

$$(2.2) \quad d_{3n} = 2c_n, \quad d_{3n+1} = c_{n+1}, \quad d_{3n+2} = c_n.$$

The Hankel matrices of this difference sequences \mathbf{d} is denoted by Δ_n^p . An easy observation show that

$$\Delta_n^p = \Gamma_n^p + \Gamma_n^{p+2}.$$

Let $K_n^p := (u_{p+3(i+j-2)})_{1 \leq i, j \leq n}$. When $\mathbf{u} = \mathbf{c}$ is the Cantor sequence, by (2.1), we have for all $n \geq 1, p \geq 0$,

$$(2.3) \quad K_n^{3p} = \Gamma_n^p, \quad K_n^{3p+1} = \mathbf{0}_{n \times n}, \quad K_n^{3p+2} = \Gamma_n^p.$$

When $\mathbf{u} = \mathbf{d}$, by (2.2), for all $n \geq 1, p \geq 0$,

$$(2.4) \quad K_n^{3p} = 2\Gamma_n^p \equiv -\Gamma_n^p, \quad K_n^{3p+1} = \Gamma_n^{p+1}, \quad K_n^{3p+2} = \Gamma_n^p,$$

where the symbol \equiv , unless otherwise stated, means equality modulo 3 throughout this paper.

Now we will give an auxiliary lemma on matrices.

Lemma 1. For all $n \geq 2, p \geq 0$,

$$\begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix} = (-1)^n |\Gamma_n^p| \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p|.$$

Proof. Suppose α_p^n is the column vector of the form $(c_p, c_{p+1}, \dots, c_{p+n-1})^t$, then

$$\begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix} = \begin{vmatrix} \Gamma_n^p & \alpha_{p+n}^n & \alpha_p^n & \Gamma_n^{p+1} \\ \mathbf{0}_{1 \times n} & 1 & 0 & \mathbf{0}_{1 \times n} \\ \mathbf{0}_{1 \times n} & 0 & 1 & \mathbf{0}_{1 \times n} \\ \Gamma_n^{p+1} & \alpha_{p+n+1}^n & \mathbf{0}_{n \times 1} & -\Gamma_n^p \end{vmatrix}.$$

Note that $(\Gamma_n^p \ \alpha_{p+n}^n) = (\alpha_p^n \ \Gamma_n^{p+1})$, we have

$$\begin{aligned} \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix} &= \begin{vmatrix} \Gamma_n^p & \alpha_{p+n}^n & 0 & \mathbf{0}_{n \times n} \\ \mathbf{0}_{1 \times n} & 1 & 0 & 0 \cdots 0 (-1) \\ \mathbf{0}_{1 \times n} & 0 & 1 & 0 \cdots 0 0 \\ \Gamma_n^{p+1} & \alpha_{p+n+1}^n & -\alpha_{p+1}^n & -\Delta_n^p \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \Gamma_n^p & \mathbf{0} \\ \Gamma_n^{p+1} & -\Delta_n^p \end{vmatrix} \\ &\quad + (-1)^{n+1} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} & \mathbf{0}_{n \times (n-1)} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & -(\Delta_n^p)^{(n)} \end{vmatrix}. \end{aligned}$$

Since

$$\begin{aligned} \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} & \mathbf{0}_{n \times (n-1)} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & -(\Delta_n^p)^{(n)} \end{vmatrix} &= \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} & \mathbf{0}_{n \times (n-1)} \\ \mathbf{0}_{(n-1) \times n} & -(\Delta_n^p)^{(n)} \\ (\alpha_{p+n+1}^n)^t & \end{vmatrix} \\ &= (-1)^{(n+1)} |\Gamma_{n+1}^p| \cdot |-\Delta_{n-1}^p| \\ &= |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p|, \end{aligned}$$

then

$$\begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix} = (-1)^n |\Gamma_n^p| \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p|.$$

□

Let

$$P(n) = (e_1, e_4, \dots, e_{3n_1-2}, e_2, e_5, \dots, e_{3n_2-1}, e_3, e_6, \dots, e_{3n_3}),$$

where $n_1 = [\frac{n+2}{3}]$, $n_2 = [\frac{n+1}{3}]$, $n_3 = [\frac{n}{3}]$ and e_j is the j -th unit column vector of order n , i.e., the column vector with 1 as the j -th entry and zeros elsewhere. And $|P(n)| = \pm 1$. For simplicity, we write P instead of $P(n)$, when no confusion can occur. When consider $P(3n)$, $P(3n+1)$, $P(3n+2)$, the following diagram shows n_1, n_2 and n_3 in these cases:

	n_1	n_2	n_3
$3n$	n	n	n
$3n+1$	$n+1$	n	n
$3n+2$	$n+1$	$n+1$	n

Suppose $M = (m_{i,j})_{1 \leq i,j \leq n}$ is an $n \times n$ matrix, then

$$P^t M P = \begin{pmatrix} (m_{3i-2,3j-2})_{n_1 \times n_1} & (m_{3i-2,3j-1})_{n_1 \times n_2} & (m_{3i-2,3j})_{n_1 \times n_3} \\ (m_{3i-1,3j-2})_{n_2 \times n_1} & (m_{3i-1,3j-1})_{n_2 \times n_2} & (m_{3i-1,3j})_{n_2 \times n_3} \\ (m_{3i,3j-2})_{n_3 \times n_1} & (m_{3i,3j-1})_{n_3 \times n_2} & (m_{3i,3j})_{n_3 \times n_3} \end{pmatrix},$$

where $(m_{3i-2,3j-1})_{s \times t}$ means the matrix $(m_{3i-2,3j-1})_{1 \leq i \leq s, 1 \leq j \leq t}$.

When $M = H_{3n}^p$, H_{3n+1}^p and H_{3n+2}^p , we have

$$\begin{aligned}
 & P^t H_{3n}^p P \\
 &= \begin{pmatrix} (U_{p+3(i+j-2)})_{n \times n} & (U_{p+3(i+j-2)+1})_{n \times n} & (U_{p+3(i+j-2)+2})_{n \times n} \\ (U_{p+3(i+j-2)+1})_{n \times n} & (U_{p+3(i+j-2)+2})_{n \times n} & (U_{p+3(i+j-2)+3})_{n \times n} \\ (U_{p+3(i+j-2)+2})_{n \times n} & (U_{p+3(i+j-2)+3})_{n \times n} & (U_{p+3(i+j-2)+4})_{n \times n} \end{pmatrix} \\
 (2.5) \quad &= \begin{pmatrix} K_n^p & K_n^{p+1} & K_n^{p+2} \\ K_n^{p+1} & K_n^{p+2} & K_n^{p+3} \\ K_n^{p+2} & K_n^{p+3} & K_n^{p+4} \end{pmatrix},
 \end{aligned}$$

$$(2.6) \quad P^t H_{3n+1}^p P = \begin{pmatrix} K_{n+1}^p & (K_{n+1}^{p+1})^{(n+1)} & (K_{n+1}^{p+2})^{(n+1)} \\ (K_{n+1}^{p+1})^{(n+1)} & K_n^{p+2} & K_n^{p+3} \\ (K_{n+1}^{p+2})^{(n+1)} & K_n^{p+3} & K_n^{p+4} \end{pmatrix},$$

$$(2.7) \quad P^t H_{3n+2}^p P = \begin{pmatrix} K_{n+1}^p & K_{n+1}^{p+1} & (K_{n+1}^{p+2})^{(n+1)} \\ K_{n+1}^{p+1} & K_{n+1}^{p+2} & (K_{n+1}^{p+3})^{(n+1)} \\ (K_{n+1}^{p+2})^{(n+1)} & (K_{n+1}^{p+3})^{(n+1)} & K_n^{p+4} \end{pmatrix},$$

3. RECURRENT EQUATIONS

In this section, we establish the recurrence formulae for the sequence $|\Gamma_n^p| (n \geq 2, p \geq 0)$, which is the key result in this paper. Through these eighteen recurrence formulae, we can evaluate all the Hankel determinants $|\Gamma_n^p|, |\Delta_n^p| (n \geq 2, p \geq 0)$.

Theorem 1. *For $p \geq 0$ and $n \geq 2$, one has*

- (1) $|\Gamma_{3n}^{3p}| = (-1)^n |\Gamma_n^p|^2 \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_n^p| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p|,$
- (2) $|\Gamma_{3n+1}^{3p}| = (-1)^n |\Gamma_n^p| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Delta_{n-1}^p|,$
- (3) $|\Gamma_{3n+2}^{3p}| = (-1)^n |\Gamma_{n+1}^p|^2 \cdot |\Delta_n^p|,$
- (4) $|\Gamma_{3n}^{3p+1}| = (-1)^n |\Gamma_n^p| \cdot |\Gamma_{n+1}^{p+1}| \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_n^{p+1}| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p|,$
- (5) $|\Gamma_{3n+1}^{3p+1}| = (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Delta_{n-1}^{p+1}|,$
- (6) $|\Gamma_{3n+2}^{3p+1}| = (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Delta_n^{p+1}|,$
- (7) $|\Gamma_{3n}^{3p+2}| = (-1)^n |\Gamma_n^{p+1}|^2 \cdot |\Delta_n^p|,$
- (8) $|\Gamma_{3n+1}^{3p+2}| = (-1)^n |\Gamma_n^{p+1}| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_n^{p+1}| + (-1)^{n+1} |\Gamma_{n+1}^{p+1}| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^{p+1}|,$
- (9) $|\Gamma_{3n+2}^{3p+2}| = (-1)^{n+1} |\Gamma_{n+1}^{p+1}|^2 \cdot |\Delta_n^p|,$
- (10) $|\Delta_{3n}^{3p}| \equiv (-1)^n |\Gamma_n^p| \cdot |\Delta_n^p|^2 + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p| \cdot |\Delta_n^p|,$
- (11) $|\Delta_{3n+1}^{3p}| \equiv (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_n^p|^2,$
- (12) $|\Delta_{3n+2}^{3p}| \equiv (-1)^n |\Gamma_{n+2}^p| \cdot |\Delta_n^p|^2 + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_n^p| \cdot |\Delta_{n+1}^p|,$
- (13) $|\Delta_{3n}^{3p+1}| \equiv (-1)^n |\Gamma_n^{p+1}| \cdot |\Delta_n^p|^2,$
- (14) $|\Delta_{3n+1}^{3p+1}| \equiv (-1)^n |\Gamma_{n+1}^{p+1}| \cdot |\Delta_n^p|^2,$
- (15) $|\Delta_{3n+2}^{3p+1}| \equiv (-1)^n |\Gamma_{n+2}^p| \cdot |\Delta_n^p| \cdot |\Delta_n^{p+1}| + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_n^{p+1}| \cdot |\Delta_{n+1}^p|,$
- (16) $|\Delta_{3n}^{3p+2}| \equiv (-1)^n |\Gamma_n^{p+1}| \cdot |\Delta_n^p| \cdot |\Delta_n^{p+1}| + (-1)^{n+1} |\Gamma_{n+1}^{p+1}| \cdot |\Delta_n^p| \cdot |\Delta_{n-1}^{p+1}|,$
- (17) $|\Delta_{3n+1}^{3p+2}| \equiv (-1)^n |\Gamma_{n+1}^p| \cdot |\Delta_n^{p+1}|^2,$
- (18) $|\Delta_{3n+2}^{3p+2}| \equiv (-1)^n |\Gamma_{n+2}^p| \cdot |\Delta_n^{p+1}|^2.$

Proof. 1) Combine (2.3) and (2.5), we have

$$\begin{aligned} |P^t \Gamma_{3n}^{3p} P| &= \begin{vmatrix} \Gamma_n^p & \mathbf{0}_{n \times n} & \Gamma_n^p \\ \mathbf{0}_{n \times n} & \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^p & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_n^p & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \Gamma_n^p & \Gamma_n^{p+1} \\ \mathbf{0}_{n \times n} & \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix}. \end{aligned}$$

Hence

$$|\Gamma_{3n}^{3p}| = |P^t \Gamma_{3n}^{3p} P| = |\Gamma_n^p| \cdot \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix}.$$

By Lemma 1,

$$|\Gamma_{3n}^{3p}| = (-1)^n |\Gamma_n^p|^2 \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_n^p| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p|.$$

2) Combine (2.3) and (2.6), we have

$$\begin{aligned} |P^t \Gamma_{3n+1}^{3p} P| &= \begin{vmatrix} \Gamma_{n+1}^p & \mathbf{0}_{n+1 \times n} & (\Gamma_{n+1}^p)^{(n+1)} \\ \mathbf{0}_{n \times n+1} & \Gamma_n^p & \Gamma_n^{p+1} \\ (\Gamma_{n+1}^p)^{(n+1)} & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & \mathbf{0}_{n+1 \times n} & \mathbf{0}_{n+1 \times n} \\ \mathbf{0}_{n \times n+1} & \Gamma_n^p & \Gamma_n^{p+1} \\ \mathbf{0}_{n \times n+1} & \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix}. \end{aligned}$$

By Lemma 1,

$$\begin{aligned} |\Gamma_{3n+1}^{3p}| &= |\Gamma_{n+1}^p| \cdot \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix} \\ &= (-1)^n |\Gamma_n^p| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Delta_{n-1}^p|. \end{aligned}$$

3) Combine (2.3) and (2.7), we have

$$\begin{aligned} |P^t \Gamma_{3n+2}^{3p} P| &= \begin{vmatrix} \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times (n+1)} & (\Gamma_{n+1}^p)^{(n+1)} \\ \mathbf{0}_{(n+1) \times (n+1)} & \Gamma_{n+1}^p & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ (\Gamma_{n+1}^p)^{(n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{n \times n} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times (n+1)} & \mathbf{0}_{(n+1) \times n} \\ \mathbf{0}_{(n+1) \times (n+1)} & \Gamma_{n+1}^p & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ (\Gamma_{n+1}^p)^{(n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} & -\Gamma_n^p \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times (n+1)} & \mathbf{0}_{(n+1) \times n} \\ \mathbf{0}_{(n+1) \times (n+1)} & \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times n} \\ (\Gamma_{n+1}^p)^{(n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} & -\Gamma_n^p - \Gamma_n^{p+2} \end{vmatrix}. \end{aligned}$$

Therefore,

$$|\Gamma_{3n+2}^{3p}| = |\Gamma_{n+1}^p|^2 \cdot |-\Gamma_n^p - \Gamma_n^{p+2}| = (-1)^n |\Gamma_{n+1}^p|^2 \cdot |\Delta_n^p|.$$

Formulae (4) to (18) can be proved using similar computation. We state the proof in the appendix. \square

Now, we will extend those eighteen recurrent formulae for all $n, p \geq 0$.

Proposition 1. Define $|\Delta_0^p| = 1$ for $p \geq 0$, and

$$|\Gamma_0^p| = \begin{cases} 2 & \text{if } p = 0 \\ 1 & \text{if } p \geq 1 \end{cases}, |\Delta_{-1}^p| = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p \geq 1 \end{cases}.$$

Then formulae of Theorem 1 holds for $p \geq 0$ and $n \geq 0$.

Proof. Using (2.1), (2.2) and the facts: $|\Gamma_1^p| = c_p$, and $|\Delta_1^p| = d_p$. We can check the formulae of Theorem 1 one by one. \square

4. PERIODICITY AND AUTOMATICITY PROPERTIES.

The periodicity and automaticity properties of the Hankel determinants $|\Gamma_n^p|$, $|\Delta_n^p|$ ($n, p \geq 0$) are discussed in this section. By the recurrent formulae in Theorem 1 and Proposition 1, to determine the quantities $\{|\Gamma_n^p|\}_{n \geq 0, p \geq 0}$, $\{|\Delta_n^p|\}_{n \geq 0, p \geq 0}$, we only need to determine the quantities for $p = 0$ and 1. The following two propositions are devoted to this purpose.

Proposition 2. With the above notation, we have

$$(4.1) \quad |\Gamma_n^0| \equiv \begin{cases} 1 & \text{if } n \equiv 1, 2 \pmod{4} \\ 2 & \text{if } n \equiv 3, 0 \pmod{4} \end{cases}, |\Delta_n^0| \equiv \begin{cases} 2 & \text{if } n \equiv 1, 2 \pmod{4} \\ 1 & \text{if } n \equiv 3, 0 \pmod{4} \end{cases}.$$

Proof. We will prove these two assertion simultaneously. For $n = 0, 1, 2$, the above equalities can be check directly. Assume that the equalities hold for $n \leq k$. By the induction hypothesis, we have for all $n < k$,

$$\begin{aligned} |\Gamma_n^0|^2 &\equiv |\Delta_n^0|^2 \equiv 1, & |\Gamma_n^0| \cdot |\Delta_n^0| &\equiv 2, \\ |\Gamma_{n+1}^0| \cdot |\Delta_{n-1}^0| &\equiv 1, & |\Gamma_{n+1}^0| + |\Delta_{n-1}^0| &\equiv |\Delta_{n+1}^2|. \end{aligned}$$

Then if $n = k + 1 = 3l(l \geq 1)$, by Theorem 1 (1) and (10), we have

$$\begin{aligned} |\Gamma_n^0| &= (-1)^l |\Gamma_l^0|^2 \cdot |\Delta_l^0| + (-1)^{l+1} |\Gamma_l^0| \cdot |\Gamma_{l+1}^0| \cdot |\Delta_{l-1}^0| \\ &\equiv (-1)^l (2|\Gamma_l^0| - |\Gamma_l^0|) \\ &\equiv (-1)^l |\Gamma_l^0|, \\ |\Delta_n^0| &= (-1)^l |\Gamma_l^0| \cdot |\Delta_l^0|^2 + (-1)^{l+1} |\Gamma_{l+1}^0| \cdot |\Delta_{l-1}^0| \cdot |\Delta_l^0| \\ &\equiv (-1)^l (2|\Delta_l^0| - |\Delta_l^0|) \\ &\equiv (-1)^l |\Delta_l^0|. \end{aligned}$$

Since $n = 3l \equiv -l \pmod{4}$, the above two equalities implies (4.1).

When $n = k + 1 = 3l + 1(l \geq 1)$, by formulae (2) and (11) of Theorem 1,

$$\begin{aligned} |\Gamma_{3l+1}^0| &= (-1)^l |\Gamma_l^0| \cdot |\Gamma_{l+1}^0| \cdot |\Delta_l^0| + (-1)^{l+1} |\Gamma_{l+2}^0|^2 \cdot |\Delta_{l-1}^0| \\ &\equiv (-1)^l (2|\Gamma_{l+1}^0| - |\Delta_{l-1}^0|) \\ &\equiv (-1)^{l+1} |\Delta_{l+1}^0|, \\ |\Delta_{3l+1}^0| &= (-1)^{l+1} |\Gamma_{l+1}^0| \cdot |\Delta_l^0|^2 \\ &\equiv (-1)^{l+1} |\Gamma_{l+1}^0|. \end{aligned}$$

Note that $n = 3l + 1 \equiv 1 - l \pmod{4}$, (4.1) follows from the above two equalities.

When $n = k + 1 = 3l + 2 (l \geq 1)$, by formulae (3) and (12) of Theorem 1, we have

$$\begin{aligned} |\Gamma_{3l+2}^0| &= (-1)^l |\Gamma_{l+1}^0|^2 \cdot |\Delta_l^0| \\ &\equiv (-1)^l |\Delta_l^0|, \\ |\Delta_{3l+2}^0| &\equiv (-1)^l |\Gamma_{l+2}^0| \cdot |\Delta_l^0|^2 + (-1)^{l+1} |\Gamma_{l+1}^0| \cdot |\Delta_l^0| \cdot |\Delta_{l+1}^0| \\ &\equiv (-1)^l (|\Delta_l^0| - 2|\Delta_l^0|) = (-1)^{l+1} |\Delta_l^0|. \end{aligned}$$

These two equalities, combining with the fact $n = 3l + 2 \equiv 2 - l \pmod{4}$, lead to (4.1). Thus the assertions are proved. \square

Proposition 3. *For $p = 1$, we have*

$$(4.2) \quad |\Gamma_n^1| \equiv |\Delta_n^1| \equiv \begin{cases} 0 & \text{if } n \equiv 1, 3 \pmod{4} \\ 2 & \text{if } n \equiv 2 \pmod{4} \\ 1 & \text{if } n \equiv 0 \pmod{4} \end{cases}.$$

Proof. These two assertions will be proved simultaneously. For $n = 0, 1, 2$, the above equalities can be check directly. Assume that the equalities hold for $n \leq k$. According to 4.1, for all $n \geq 1$,

$$|\Gamma_n^0|^2 \equiv |\Delta_n^0|^2 \equiv 1, |\Gamma_n^0| \cdot |\Delta_n^0| \equiv 2, |\Gamma_{n+1}^0| \cdot |\Delta_{n-1}^0| \equiv 1.$$

By the induction hypothesis, for all $1 \leq n < k$,

$$|\Gamma_{n-1}^1| \equiv -|\Gamma_{n+1}^1|.$$

Then if $n = k + 1 = 3l (l \geq 1)$, by formulae (4) and (13) of Theorem 1, we have

$$\begin{aligned} |\Gamma_n^1| &= (-1)^l |\Gamma_l^0| \cdot |\Gamma_l^1| \cdot |\Delta_l^0| + (-1)^{l+1} |\Gamma_l^1| \cdot |\Gamma_{l+1}^0| \cdot |\Delta_{l-1}^0| \\ &\equiv (-1)^l (2|\Gamma_l^1| - |\Gamma_l^1|) \\ &\equiv (-1)^l |\Gamma_l^1|, \\ |\Delta_n^1| &\equiv (-1)^l |\Gamma_l^1| \cdot |\Delta_l^0|^2 \equiv (-1)^l |\Gamma_l^1|. \end{aligned}$$

Since $n = 3l \equiv -l \pmod{4}$, (4.2) holds in this case.

When $n = k + 1 = 3l + 1 (l \geq 1)$, by formulae (5) and (14) and Theorem 1, we have

$$\begin{aligned} |\Gamma_n^1| &= (-1)^{l+1} |\Gamma_{l+1}^0|^2 \cdot |\Delta_{l-1}^1| \equiv (-1)^{l+1} |\Delta_{l-1}^1| \equiv (-1)^l |\Gamma_{l+1}^1|, \\ |\Delta_n^1| &\equiv (-1)^l |\Gamma_{l+1}^1| \cdot |\Delta_l^0|^2 \equiv (-1)^l |\Gamma_{l+1}^1|. \end{aligned}$$

Since $n = 3l + 1 \equiv 1 - l \pmod{4}$, (4.2) holds in this case.

When $n = k + 1 = 3l + 2 (l \geq 1)$, by formulae (6) and (15) of Theorem 1, we have

$$\begin{aligned} |\Gamma_n^1| &= (-1)^{l+1} |\Gamma_{l+1}^0|^2 \cdot |\Delta_l^1| \equiv (-1)^{l+1} |\Delta_l^1|, \\ |\Delta_n^1| &\equiv (-1)^l |\Gamma_{l+2}^0| \cdot |\Delta_l^0| \cdot |\Delta_l^1| + (-1)^{l+1} |\Gamma_{l+1}^0| \cdot |\Delta_l^1| \cdot |\Delta_{l+1}^0| \\ &\equiv (-1)^l (|\Delta_l^1| - 2|\Delta_l^1|) \equiv (-1)^{l+1} |\Delta_l^1|. \end{aligned}$$

These two equalities, combining with the fact $n = 3l + 2 \equiv 2 - l \pmod{4}$, lead to (4.2). Thus the assertions are proved. \square

4.1. Periodicity properties. Let $(u_n)_{n \geq 0}$ be a sequence with $u_n \in \mathbb{F}_3$, then the formal power series

$$u(x) = \sum_{n \geq 0} u_n x^n$$

is called the *generating series* of the sequence $(u_n)_{n \geq 0}$.

A sequence $(u_n)_{n \geq 0}$ is periodic of period t if and only if its generating series adds up to a rational fraction of the form $\frac{P(x)}{1-x^t}$, where $P(x)$ is a polynomial of degree less than t .

Let $P(x) = \sum_{n \geq 0} a_n x^n$ and $Q(x) = \sum_{n \geq 0} b_n x^n$ be two formal power series with $a_n, b_n \in \mathbb{F}_3$, then their *Hadamard product* is defined to be

$$P(x) \star Q(x) = \sum_{n \geq 0} a_n b_n x^n.$$

In addition, if $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are periodic of period s and t respectively, then $P(x) \star Q(x)$ is the generating series of the periodic sequence $(a_n b_n)_{n \geq 0}$ having $[s, t]$ as a period, where $[s, t]$ denotes the lowest common multiple of s and t .

For $p \geq 0$, define

$$f^{(p)}(x) = \sum_{n \geq 0} |\Gamma_n^p| x^n, \quad g^{(p)}(x) = \sum_{n \geq 0} |D_n^p| x^n,$$

and

$$h_0(x) = \sum_{n \geq 0} (-1)^n x^n, \quad h_1(x) = \sum_{n \geq 0} (-1)^{n+1} x^n$$

where the coefficients are taken modulo 3, with the convention of Proposition 1.

By Proposition 2 and Proposition 3, we have

$$(4.3) \quad \begin{cases} f^{(0)} = \frac{2+x+x^2+2x^3}{1-x^4}, & g^{(0)} = \frac{1+2x+2x^2+x^3}{1-x^4}, \\ f^{(1)} = \frac{1+2x^2}{1-x^4}, & g^{(1)} = \frac{1+2x^2}{1-x^4}. \end{cases}$$

Using the recurrent formulae in Theorem 1, we can compute the above quantities recursively. We compute $f^{(2)}$ and $g^{(2)}$ as an example.

$$\begin{aligned} f^{(2)} &= \sum_{n \geq 0} |\Gamma_n^2| x^n = \sum_{n \geq 0} (|\Gamma_{3n}^2| + |\Gamma_{3n+1}^2| x + |\Gamma_{3n+2}^2| x^2) x^{3n} \\ &= \sum_{n \geq 0} \left[(-1)^n |\Gamma_n^1|^2 |\Delta_n^0| + ((-1)^n |\Gamma_n^1| |\Gamma_{n+1}^0| |\Delta_n^1| \right. \\ &\quad \left. + (-1)^{n+1} |\Gamma_{n+1}^0| |\Gamma_{n+1}^1| |\Delta_{n-1}^1|) x + (-1)^{n+1} |\Gamma_{n+1}^1|^2 |\Delta_n^0| x^2 \right] x^{3n} \\ &\text{by Theorem 1} \\ &= (h_0 \star f^{(1)} \star f^{(1)} \star g^{(0)})^3 + x(h_0 \star f^{(1)} \star \widehat{f^{(0)}} \star g^{(1)} \\ &\quad + h_1 \star \widehat{f^{(0)}} \star \widehat{f^{(1)}} \star \overline{g^{(1)}})^3 + x^2(h_1 \star \widehat{f^{(1)}} \star \widehat{f^{(1)}} \star g^{(0)})^3 \end{aligned}$$

where

$$(4.4) \quad h_0(x) = \sum_{n \geq 0} (-1)^n x^n, \quad h_1(x) = \sum_{n \geq 0} (-1)^{n+1} x^n$$

and

$$(4.5) \quad \widehat{f^{(1)}}(x) = \sum_{n \geq 0} |\Gamma_{n+1}^1| x^n, \quad \overline{g^{(1)}}(x) = \sum_{n \geq 0} |\Delta_{n-1}^1| x^n.$$

Thus by (4.3), (4.4) and (4.5), we have

$$\begin{aligned}
f^{(2)} &= \left(\frac{1+2x^2}{1-x^4} \right)^3 + x \left(\frac{1+2x^2}{1-x^4} + \frac{2x+x^3}{1-x^4} \right)^3 + x^2 \left(\frac{2x+x^3}{1-x^4} \right)^3 \\
&= \frac{1+x+2x^4+2x^5+2x^6+2x^7+x^{10}+x^{11}}{1-x^{12}}; \\
g^{(2)} &= \sum_{n \geq 0} |\Delta_n^2| x^n = \sum_{n \geq 0} (|\Delta_{3n}^2| + x|\Delta_{3n+1}^2| + x^2|\Delta_{3n+2}^2|) x^{3n} \\
&= \sum_{n \geq 0} \left[((-1)^n |\Gamma_n^1| |\Delta_n^0| |\Delta_n^1| + (-1)^{n+1} |\Gamma_{n+1}^1| |\Delta_n^0| |\Delta_{n-1}^1|) \right. \\
&\quad \left. + x((-1)^n |\Gamma_{n+1}^0| |\Delta_n^1|^2) + x^2((-1)^n |\Gamma_{n+2}^0| |\Delta_n^1|^2) \right] x^{3n} \\
&= (h_0 \star f^{(1)} \star g^{(0)} \star g^{(1)} + h_1 \star \widehat{f^{(1)}} \star g^{(0)} \star \overline{g^{(1)}})^3 \\
&\quad + x(h_0 \star \widehat{f^{(0)}} \star g^{(1)} \star g^{(1)})^3 + x^2(h_0 \star \widehat{\widehat{f^{(0)}}} \star g^{(1)} \star g^{(1)})^3
\end{aligned}$$

where

$$(4.6) \quad \widehat{\widehat{f^{(0)}}}(x) = \sum_{n \geq 0} |\Gamma_{n+2}^0| x^n.$$

Thus by (4.3), (4.4), (4.5) and (4.6), we have

$$g^{(2)} = \frac{1+x+x^2+2x^6+2x^7+2x^8}{1-x^{12}}.$$

Theorem 2. For any $p \geq 0$, the sequences (modulo 3)

$$\{|\Gamma_n^p|\}_{n \geq 0}, \{|\Delta_n^p|\}_{n \geq 0}$$

are both periodic. Moreover, $12 \cdot 3^k$ is a period if $3^k + 1 \leq p \leq 3^{k+1}$.

Proof. For $p = 0, 1, 2, 3$ by the recurrence formulae of Proposition 1 and equalities (4.3) these two sequences are periodic. Now suppose $p \geq 4$, we shall prove by induction on k that $12 \cdot 3^k$ is a period if $3^k + 1 \leq p \leq 3^{k+1}$.

By calculation, we can find that the conclusion is true for $k = 1$. Suppose that the conclusion is true for $p \leq 3^k$. We need to show that the conclusion is true for $3^k + 1 \leq p \leq 3^{k+1}$. If $p = 3q$, then $3^{k-1} + 1 \leq q \leq 3^k$, thus by Theorem 1 (1) (2) and (3), we have

$$\begin{aligned}
|\Gamma_{3n}^p| &= (-1)^n |\Gamma_n^q|^2 \cdot |\Delta_n^q| + (-1)^{n+1} |\Gamma_n^q| \cdot |\Gamma_{n+1}^q| \cdot |\Delta_{n-1}^q|, \\
|\Gamma_{3n+1}^p| &= (-1)^n |\Gamma_n^q| \cdot |\Gamma_{n+1}^q| \cdot |\Delta_n^q| + (-1)^{n+1} |\Gamma_{n+2}^q|^2 \cdot |\Delta_{n-1}^q|, \\
|\Gamma_{3n+2}^p| &= (-1)^n |\Gamma_{n+1}^q|^2 \cdot |\Delta_n^q|.
\end{aligned}
\tag{4.7}$$

By the induction hypothesis, all the sequences occurring on the right hand of (4.7) have period $12 \cdot 3^{k-1}$, and so do the product and sum of these sequences. Therefore, the sequences $|\Gamma_{3n}^p|$, $|\Gamma_{3n+1}^p|$ and $|\Gamma_{3n+2}^p|$ are all $12 \cdot 3^{k-1}$ -periodic which implies that the sequence $|\Gamma_n^p|$ is of period $12 \cdot 3^k$. The cases $p = 3q + 1$ and $3q + 2$ follow in the same way. Similar discussions can be applied to the sequence $|\Delta_n^p|$. \square

4.2. Automaticity properties. First, we will recall some definitions of two dimensional automatic sequences which can be found in [2, Chapter 14].

Let \mathcal{A}, \mathcal{B} be two finite alphabets. If

$$A = (a_{i,j})_{0 \leq i \leq m, 0 \leq j \leq n}$$

is an $m \times n$ matrix with entries in \mathcal{A} , and $\psi : \mathcal{A} \rightarrow \mathcal{B}^{k \times l}$ is a $[k, l]$ -uniform matrix-valued morphism, i.e., a map sending each letter in \mathcal{A} to an $k \times l$ matrix, then $\psi(A)$ is an $km \times ln$ matrix given by

$$\begin{bmatrix} \psi(a_{00}) & \psi(a_{01}) & \cdots & \psi(a_{0,n-1}) \\ \psi(a_{10}) & \psi(a_{11}) & \cdots & \psi(a_{1,n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi(a_{m-1,0}) & \psi(a_{m-1,1}) & \cdots & \psi(a_{m-1,n-1}) \end{bmatrix}.$$

A $[k, l]$ -automatic sequence is the image (under a coding) of a fixed point of a $[k, l]$ -morphism. In particular, if $k = l$, the $[k, k]$ -automatic sequence is also called the k -automatic sequence. A well known result [2, Theorem 14.2.2] (see also [9], [10]) shows that the two-dimensional sequence $\mathbf{u} = (u_{n,m})_{n,m \geq 0}$ is $[k, l]$ -automatic sequences if and only if the $[k, l]$ -kernel $K_{k,l}(\mathbf{u})$ is finite, where

$$K_{k,l}(\mathbf{u}) = \{(u_{k^a m+r, l^a n+s})_{m,n \geq 0} : a \geq 0, 0 \leq r < k^a, 0 \leq s < l^a\}.$$

Theorem 3. *The two-dimensional sequences (modulo 3)*

$$\{|\Gamma_n^p|\}_{n \geq 0, p \geq 0}, \{|\Delta_n^p|\}_{n \geq 0, p \geq 0}$$

are both 3-automatic.

Proof. For this, we only need to show that the 3-kernels of these sequences are finite.

Let $\{u_n^p\}_{n \geq -1, p \geq 0}$ be a double sequence. For $\alpha' \in \{-1, 0, 1, 2\}$ and $\alpha, \beta, \beta' \in \{0, 1, 2\}$, operations $S_{\alpha'}^{\beta'}$ and T_{α}^{β} are defined as follows

$$S_{\alpha'}^{\beta'} u = \{u_{n+\alpha'}^{p+\beta'}\}_{n \geq 0, p \geq 0}, \quad T_{\alpha}^{\beta} u = \{u_{3n+\alpha}^{3p+\beta}\}_{n \geq 0, p \geq 0}.$$

Then, for $\alpha' \in \{-1, 0, 1, 2\}$ and $\alpha, \beta, \beta' \in \{0, 1, 2\}$, we have

$$(4.8) \quad T_{\alpha}^{\beta} S_{\alpha'}^{\beta'} = \begin{cases} S_{-1}^0 T_2^{\beta+\beta'} & \text{if } \alpha + \alpha' = -1 \text{ and } \beta + \beta' \leq 2, \\ S_{-1}^1 T_2^{\beta+\beta'-3} & \text{if } \alpha + \alpha' = -1 \text{ and } \beta + \beta' \geq 3, \\ T_{\alpha+\alpha'}^{\beta+\beta'} & \text{if } 0 \leq \alpha + \alpha' \leq 2 \text{ and } \beta + \beta' \leq 2, \\ S_0^1 T_{\alpha+\alpha'}^{\beta+\beta'-3} & \text{if } 0 \leq \alpha + \alpha' \leq 2 \text{ and } \beta + \beta' \geq 3, \\ S_1^0 T_{\alpha+\alpha'}^{\beta+\beta'} & \text{if } \alpha + \alpha' \geq 3 \text{ and } \beta + \beta' \leq 2, \\ S_1^1 T_{\alpha+\alpha'}^{\beta+\beta'-3} & \text{if } \alpha + \alpha' \geq 3 \text{ and } \beta + \beta' \geq 3. \end{cases}$$

Suppose Γ, Δ and F stand for the sequences $\{|\Gamma_n^p|\}_{n \geq 0, p \geq 0}, \{|\Delta_n^p|\}_{n \geq 0, p \geq 0}$ and $\{F_n^p\}_{n \geq 0, p \geq 0}$ modulo 3 where $F_n^p = (-1)^n$. Thus for any $\beta \in \{0, 1, 2\}$

$$(4.9) \quad T_0^{\beta} F = T_2^{\beta} F = S_2^{\beta} F = F \text{ and } T_1^{\beta} F = S_1^{\beta} F = S_1^0 F.$$

We rewrite Theorem 1 and Proposition 1 as follows

$$(4.10) \quad \left\{ \begin{array}{ll} T_0^0 \Gamma \equiv F \cdot \Gamma^2 \cdot \Delta & T_0^2 \Gamma \equiv F \cdot (S_0^1 \Gamma)^2 \cdot \Delta, \\ \quad + S_1^0 F \cdot \Gamma \cdot S_1^0 \Gamma \cdot S_{-1}^0 \Delta, & \\ T_0^0 \Delta \equiv F \cdot \Gamma \cdot \Delta^2 & T_0^2 \Delta \equiv F \cdot S_0^1 \Gamma \cdot \Delta \cdot S_0^1 \Delta \\ \quad + S_1^0 F \cdot S_1^0 \Gamma \cdot S_{-1}^0 \Delta \cdot \Delta, & \quad + S_1^0 F \cdot S_1^1 \Gamma \cdot \Delta \cdot S_{-1}^1 \Delta, \\ T_1^0 \Gamma \equiv F \cdot S_1^0 \Gamma \cdot \Gamma \cdot \Delta & T_1^2 \Gamma \equiv F \cdot S_1^0 \Gamma \cdot S_0^1 \Gamma \cdot S_0^1 \Delta \\ \quad + S_1^0 F \cdot (S_1^0 \Gamma)^2 \cdot S_{-1}^0 \Delta, & \quad + S_1^0 F \cdot S_1^0 \Gamma \cdot S_1^1 \Gamma \cdot S_{-1}^1 \Delta, \\ T_1^0 \Delta \equiv S_1^0 F \cdot S_1^0 \Gamma \cdot \Delta^2, & T_1^2 \Delta \equiv F \cdot S_1^0 \Gamma \cdot (S_0^1 \Delta)^2, \\ T_2^0 \Gamma \equiv F \cdot (S_1^0 \Gamma)^2 \cdot \Delta, & T_2^2 \Gamma \equiv S_1^0 F \cdot (S_1^1 \Gamma)^2 \cdot \Delta, \\ T_2^0 \Delta \equiv F \cdot S_2^0 \Gamma \cdot \Delta^2 & T_2^2 \Delta \equiv F \cdot S_2^0 \Gamma \cdot (S_0^1 \Delta)^2, \\ \quad + S_1^0 F \cdot S_1^0 \Gamma \cdot \Delta \cdot S_1^0 \Delta, & \\ T_0^1 \Gamma \equiv F \cdot S_0^1 \Gamma \cdot \Gamma \cdot \Delta & T_2^1 \Gamma \equiv S_1^0 F \cdot (S_1^0 \Gamma)^2 \cdot S_0^1 \Delta, \\ \quad + S_1^0 F \cdot S_1^0 \Gamma \cdot S_0^1 \Gamma \cdot S_{-1}^0 \Delta, & \\ T_0^1 \Delta \equiv F \cdot S_0^1 \Gamma \cdot \Delta^2, & T_2^1 \Delta \equiv F \cdot S_2^0 \Gamma \cdot \Delta \cdot S_0^1 \Delta \\ \quad + S_1^0 F \cdot S_1^0 \Gamma \cdot S_0^1 \Delta \cdot S_1^0 \Delta, & \\ T_1^1 \Gamma \equiv S_1^0 F \cdot (S_1^0 \Gamma)^2 \cdot S_{-1}^1 \Delta, & T_1^1 \Delta \equiv F \cdot S_1^1 \Gamma \cdot \Delta^2. \end{array} \right.$$

Let $\mathcal{X} = \{\Gamma, \Delta, F\}$ and $\mathcal{Y} = \{S_\alpha^\beta J \mid J \in \mathcal{X}, \alpha = -1, 0, 1, 2 \text{ and } \beta = 0, 1, 2\}$. According to (4.8) (4.9) and (4.10), for any $\alpha, \beta \in \{0, 1, 2\}$ and $J \in \mathcal{Y}$, $T_\alpha^\beta J$ can be expressed as a polynomial with coefficients in GF_3 of the elements of \mathcal{Y} . Hence the elements of 3-kernels $K_3(\Gamma)$ and $K_3(\Delta)$ are obtained by successive applications of operators T_α^β . For instance, let $(|\Gamma_{3^m n+r}^{3^m p+s}|)_{n,p \geq 0} \in K_3(\Gamma)$ where $m \geq 0$ and $0 \leq r, s \leq 3^m$. Suppose $r = \sum_{i=0}^{m-1} 3^i \alpha_i$ and $s = \sum_{i=0}^{m-1} 3^i \beta_i$ where $\alpha_i, \beta_i \in \{0, 1, 2\}$. It is easy to verify that $(|\Gamma_{3^m n+r}^{3^m p+s}|)_{n,p \geq 0} = T_{\alpha_{m-1}}^{\beta_{m-1}} \cdots T_{\alpha_1}^{\beta_1} T_{\alpha_0}^{\beta_0}(\Gamma)$. Therefore the elements of these two 3-kernels are polynomials with coefficients in GF_3 of the elements of \mathcal{Y} .

Since there is only a finite number of polynomial functions on GF_3 with twelve variables, these 3-kernels are finite. Therefore, the sequences $\{|\Gamma_n^p|\}_{n \geq 0, p \geq 0}$ and $\{|\Delta_n^p|\}_{n \geq 0, p \geq 0}$ are 3-automatic. \square

Corollary 1. *For any $n \geq 1$, the sequences (modulo 3)*

$$\{|\Gamma_n^p|\}_{p \geq 0}, \{|\Delta_n^p|\}_{p \geq 0}$$

are both 3-automatic.

Proof. An immediate consequence of Salon [9], [10] is that, if a two-dimensional sequence $\{s_{m,n}\}_{m,n \geq 0}$ is 3-automatic, then for any fixed $m \geq 0$ the sequence $\{s_{m,n}\}_{n \geq 0}$ is 3-automatic which prove our result. \square

5. APPLICATIONS.

5.1. Padé approximation. Now, consider again the Cantor sequence

$$\mathbf{c} = c_0 c_1 c_2 \cdots \in \{0, 1\}^{\mathbb{N}},$$

let

$$f(x) = \sum_{n \geq 0} c_n x^n$$

be the generating function of the Cantor sequence. It follows from (2.1) that

$$(5.1) \quad f(x) = (1 + x^2)f(x^3),$$

and $f(x) > 1$ for any $x > 0$.

Denote by $\left[\frac{p}{q}\right]_f$, a (p, q) -order Padé approximate of f , i.e., a rational function $P(x)/Q(x)$ whose denominator has degree q and whose numerator has degree p such that

$$f(x) - \frac{P(x)}{Q(x)} = O(x^{p+q+1}), \quad x \rightarrow 0.$$

A classical result [4, Brezinski, Page 35], shows us that if $\Gamma_n^0 \neq 0$, then the Padé approximate $\left[\frac{n-1}{n}\right]_f$ exists. Moreover,

$$(5.2) \quad f(x) - [n-1/n]_f(x) = \frac{\Gamma_{n+1}^0}{\Gamma_n^0} x^{2n} + O(x^{2n+1}).$$

Hence by Proposition 2, we have the following theorem.

Theorem 4. *Let $f(x) = \sum_{n \geq 0} c_n x^n$, then for any $n \geq 1$, the $(n-1, n)$ -order Padé approximate of f exists.*

5.2. The irrationality exponent of the Cantor number. Let ξ be an irrational number. The *irrationality exponent* (or *irrational measure*) $\mu(\xi)$ of ξ is defined as follow

$$\mu(\xi) = \sup \left\{ \mu \in \mathbb{R} : \left| \xi - \frac{p}{q} \right| < \frac{1}{q^\mu} \text{ for infinite many } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

Let $\xi_{\mathbf{c}, b}$ be the *Cantor number* defined by

$$\xi_{\mathbf{c}, b} = \sum_{k \geq 0} \frac{c_k}{b^k} = 1 + \frac{1}{b^2} + \frac{1}{b^6} + \frac{1}{b^8} + \cdots,$$

where $\mathbf{c} = c_0 c_1 c_2 \cdots$ is the Cantor sequence. Combine equation (5.1), Proposition 1 and Theorem 1 in [7], we have

Proposition 4. *For any integer $b \geq 2$, the irrationality exponent of the Cantor number $\xi_{\mathbf{c}, b}$ is equal to 2.*

Corollary 2. *For any $b \geq 2$, let $\eta_{\mathbf{d}, b} = \sum_{n \geq 0} d_n b^{-n}$. The irrationality exponent of $\eta_{\mathbf{d}, b}$ is equal to 2.*

Proof. Using (2.2), we have

$$\begin{aligned} \eta_{\mathbf{d}, b} &= \sum_{n \geq 0} d_n b^{-n} = \sum_{n \geq 0} (d_{3n} + d_{3n+1} b^{-1} + d_{3n+2} b^{-2}) b^{-3n} \\ &= \sum_{n \geq 0} (2c_n + c_{n+1} b^{-1} + c_n b^{-2}) b^{-3n} \\ &= 2\xi_{\mathbf{c}, b^3} + b^{-1}(\xi_{\mathbf{c}, b^3} - 1) + b^{-2}\xi_{\mathbf{c}, b^3} \\ &= (2 + b^{-1} + b^{-2})\xi_{\mathbf{c}, b^3} - b^{-1}. \end{aligned}$$

Since the irrationality exponent is invariant under multiplication and addition of a rational number, we can deduce from Proposition 4 that the irrationality exponent of $\eta_{\mathbf{d}, b}$ is equal to 2. \square

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APPENDIX A. PROOF OF THEOREM 1 : CONTINUE.

Proof. 4) Combine (2.3) and (2.5), we have

$$\begin{aligned}
 |P^t \Gamma_{3n}^{3p+1} P| &= \begin{vmatrix} \mathbf{0}_{n \times n} & \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^p & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ \Gamma_n^{p+1} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \end{vmatrix} \\
 &= \begin{vmatrix} -\Gamma_n^{p+1} & \Gamma_n^p & \mathbf{0}_{n \times n} \\ \Gamma_n^p & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \end{vmatrix}.
 \end{aligned}$$

Since

$$\begin{vmatrix} -\Gamma_n^{p+1} & \Gamma_n^p \\ \Gamma_n^p & \Gamma_n^{p+1} \end{vmatrix} = \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix},$$

by Lemma 1,

$$\begin{aligned}
 |\Gamma_{3n}^{3p+1}| &= |\Gamma_n^{p+1}| \cdot \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^p \end{vmatrix} \\
 &= (-1)^n |\Gamma_n^{p+1}| \cdot |\Gamma_n^p| \cdot |\Delta_n^p| + (-1)^{n+1} |\Gamma_n^{p+1}| \cdot |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p|.
 \end{aligned}$$

5) Combine (2.3) and (2.6), we have

$$|P^t \Gamma_{3n+1}^{3p+1} P| = \begin{vmatrix} \mathbf{0}_{(n+1) \times (n+1)} & (\Gamma_{n+1}^p)^{(n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ (\Gamma_{n+1}^p)^{(n+1)} & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \end{vmatrix}.$$

Recall that α_p^n is the column vector of the form $(c_p, c_{p+1}, \dots, c_{p+n-1})^t$, then

$$\begin{aligned}
|P^t \Gamma_{3n+1}^{3p+1} P| &= \begin{vmatrix} \mathbf{0}_{(n+1) \times (n+1)} & (\Gamma_{n+1}^p)^{(n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ (\Gamma_{n+1}^p)^{(n+1)} & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{(n-1) \times (n+1)} & -(\Gamma_n^{p+1})_{(1)} & (\Gamma_n^{p+1})_{(n)} \\ (\alpha_{p+n}^{n+1})^t & \mathbf{0}_{1 \times n} & (\alpha_{p+n}^n)^t \end{vmatrix} \\
&= \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} \\ (\alpha_{p+n}^{n+1})^t \end{vmatrix} \cdot \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ -(\Gamma_n^{p+1})_{(1)} & (\Gamma_n^{p+1})_{(n)} \end{vmatrix} \\
&= |\Gamma_{n+1}^p| \cdot \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ -(\Gamma_n^{p+1})_{(1)} & (\Gamma_n^{p+1})_{(n)} \end{vmatrix} \\
&= |\Gamma_{n+1}^p| \cdot \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} & \mathbf{0}_{(n+1) \times (n-1)} & \alpha_{p+n}^{n+1} \\ -(\Gamma_n^{p+1})_{(1)} & \Gamma_{n-1}^{p+1} + \Gamma_{n-1}^{p+3} & \alpha_{p+n}^{n-1} \end{vmatrix} \\
&= |\Gamma_{n+1}^p| \cdot (-1)^{n+1} \cdot \begin{vmatrix} (\Gamma_{n+1}^p)^{(n+1)} & \alpha_{p+n}^{n+1} \\ (\Gamma_n^{p+1})_{(1)} & \Gamma_{n-1}^{p+1} + \Gamma_{n-1}^{p+3} \end{vmatrix} \\
&= (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Gamma_{n-1}^{p+1} + \Gamma_{n-1}^{p+3}|.
\end{aligned}$$

Therefore,

$$|\Gamma_{3n+1}^{3p+1}| = (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Delta_{n-1}^{p+1}|.$$

6) Combine (2.3) and (2.7), we have

$$\begin{aligned}
|P^t \Gamma_{3n+2}^{3p+1} P| &= \begin{vmatrix} \mathbf{0}_{(n+1) \times (n+1)} & \Gamma_{n+1}^p & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ \Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} & \mathbf{0}_{(n+1) \times n} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{n \times (n+1)} & \Gamma_n^{p+1} \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{0}_{(n+1) \times (n+1)} & \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times n} \\ \Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} & -(\Gamma_{n+1}^{p+1})_{(1)} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{n \times (n+1)} & \Gamma_n^{p+1} \end{vmatrix} \\
&= (-1)^{n+1} |\Gamma_{n+1}^p| \cdot \begin{vmatrix} \Gamma_{n+1}^p & -(\Gamma_{n+1}^{p+1})_{(1)} \\ (\Gamma_{n+1}^{p+1})_{(n+1)} & \Gamma_n^{p+1} \end{vmatrix} \\
&= (-1)^{n+1} |\Gamma_{n+1}^p| \cdot \begin{vmatrix} \Gamma_{n+1}^p & -(\Gamma_{n+1}^{p+1})_{(1)} \\ \mathbf{0}_{n \times (n+1)} & \Gamma_n^{p+1} + \Gamma_n^{p+3} \end{vmatrix} \\
&= (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Delta_n^{p+1}|.
\end{aligned}$$

Therefore,

$$|\Gamma_{3n+2}^{3p+1}| = (-1)^{n+1} |\Gamma_{n+1}^p|^2 \cdot |\Delta_n^{p+1}|.$$

7) Combine (2.3) and (2.5), we have

$$\begin{aligned}
|P^t \Gamma_{3n}^{3p+2} P| &= \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ \Gamma_n^{p+1} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \\ \mathbf{0}_{n \times n} & \Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\
&= \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \\ -\Gamma_n^{p+2} & \Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\
&= \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \\ -\Gamma_n^{p+2} - \Gamma_n^p & \mathbf{0}_{n \times n} & \Gamma_n^{p+2} \end{vmatrix} \\
&= (-1)^n |\Gamma_n^{p+1}|^2 \cdot |\Gamma_n^p + \Gamma_n^{p+2}|.
\end{aligned}$$

Therefore,

$$|\Gamma_{3n}^{3p+2}| = (-1)^n |\Gamma_n^{p+1}|^2 \cdot |\Delta_n^p|.$$

8) Combine (2.3) and (2.6), we have

$$\begin{aligned} |P^t \Gamma_{3n+1}^{3p+2} P| &= \begin{vmatrix} \Gamma_{n+1}^p & (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{(n+1) \times n} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \\ \mathbf{0}_{n \times (n+1)} & \Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times n} & \mathbf{0}_{(n+1) \times n} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & -\Gamma_n^{p+2} & \Gamma_n^{p+1} \\ \mathbf{0}_{n \times (n+1)} & \Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\ &= |\Gamma_{n+1}^p| \cdot \begin{vmatrix} -\Gamma_n^{p+2} & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\ &= |\Gamma_{n+1}^p| \cdot \begin{vmatrix} \Gamma_n^{p+1} \Gamma_n^{p+2} \\ \Gamma_n^{p+2} - \Gamma_n^{p+1} \end{vmatrix}. \end{aligned}$$

By Lemma 1,

$$|\Gamma_{3n+1}^{3p+2}| = (-1)^n |\Gamma_{n+1}^p| \cdot |\Gamma_n^{p+1}| \cdot |\Delta_n^{p+1}| + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Gamma_{n+1}^{p+1}| \cdot |\Delta_{n-1}^{p+1}|.$$

9) Combine (2.3) and (2.7), we have

$$\begin{aligned} |P^t \Gamma_{3n+2}^{3p+2} P| &= \begin{vmatrix} \Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} & \mathbf{0}_{(n+1) \times n} \\ \Gamma_{n+1}^{p+1} & \mathbf{0}_{(n+1) \times (n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} \\ \mathbf{0}_{n \times (n+1)} & (\Gamma_{n+1}^{p+1})^{(n+1)} & \Gamma_n^{p+2} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} & -(\Gamma_{n+1}^p)^{(n+1)} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{(n+1) \times (n+1)} & \mathbf{0}_{(n+1) \times n} \\ \mathbf{0}_{n \times (n+1)} & \Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} & -(\Gamma_{n+1}^p)^{(n+1)} \\ (\Gamma_{n+1}^{p+1})^{(n+1)} & \mathbf{0}_{(n+1) \times (n+1)} & \mathbf{0}_{(n+1) \times n} \\ -(\Gamma_{n+1}^p)^{(n+1)} & \mathbf{0}_{n \times (n+1)} & \Gamma_n^p + \Gamma_n^{p+2} \end{vmatrix} \\ &= (-1)^{n+1} |\Gamma_{n+1}^{p+1}| \cdot \begin{vmatrix} \Gamma_{n+1}^{p+1} & -(\Gamma_{n+1}^p)^{(n+1)} \\ \mathbf{0}_{n \times (n+1)} & \Gamma_n^p + \Gamma_n^{p+2} \end{vmatrix} \\ &= (-1)^{n+1} |\Gamma_{n+1}^{p+1}|^2 \cdot |\Gamma_n^p + \Gamma_n^{p+2}|. \end{aligned}$$

Therefore,

$$|\Gamma_{3n+2}^{3p+2}| = (-1)^{n+1} |\Gamma_{n+1}^{p+1}|^2 \cdot |\Delta_n^p|.$$

10) Combine (2.4) and (2.5), we have

$$\begin{aligned} |P^t \Delta_{3n}^{3p} P| &= \begin{vmatrix} 2\Gamma_n^p & \Gamma_n^{p+1} & \Gamma_n^p \\ \Gamma_n^{p+1} & \Gamma_n^p & 2\Gamma_n^{p+1} \\ \Gamma_n^p & 2\Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\ &\equiv \begin{vmatrix} -\Gamma_n^p & \Gamma_n^{p+1} & \Gamma_n^p \\ \Gamma_n^{p+1} & \Gamma_n^p & -\Gamma_n^{p+1} \\ \Gamma_n^p & -\Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\ &= \begin{vmatrix} -\Gamma_n^p & \Gamma_n^{p+1} & \mathbf{0}_{n \times n} \\ \Gamma_n^{p+1} & \Gamma_n^p & \mathbf{0}_{n \times n} \\ \Gamma_n^p & -\Gamma_n^{p+1} & \Gamma_n^p + \Gamma_n^{p+2} \end{vmatrix}. \end{aligned}$$

Hence, by Lemma 1,

$$\begin{aligned} |\Delta_{3n}^{3p}| &\equiv |\Gamma_n^p + \Gamma_n^{p+2}| \cdot \begin{vmatrix} -\Gamma_n^p & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & \Gamma_n^p \end{vmatrix} \\ &= (-1)^n |\Gamma_n^p| \cdot |\Delta_n^p|^2 + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_{n-1}^p| \cdot |\Delta_n^p|. \end{aligned}$$

11) Combine (2.4) and (2.6), we have

$$\begin{aligned} \left| P^t \Delta_{3n+1}^{3p} P \right| &\equiv \begin{vmatrix} -\Gamma_{n+1}^p & \left(\Gamma_{n+1}^{p+1} \right)^{(n+1)} & \left(\Gamma_{n+1}^p \right)^{(n+1)} \\ \left(\Gamma_{n+1}^{p+1} \right)_{(n+1)} & \Gamma_n^p & -\Gamma_n^{p+1} \\ \left(\Gamma_{n+1}^p \right)_{(n+1)} & -\Gamma_n^{p+1} & \Gamma_n^{p+2} \end{vmatrix} \\ &= \begin{vmatrix} -\Gamma_{n+1}^p & \left(\Gamma_{n+1}^{p+1} \right)^{(n+1)} & \mathbf{0}_{(n+1) \times n} \\ \left(\Gamma_{n+1}^{p+1} \right)_{(n+1)} & \Gamma_n^p & \mathbf{0}_{n \times n} \\ \left(\Gamma_{n+1}^p \right)_{(n+1)} & -\Gamma_n^{p+1} & \Gamma_n^p + \Gamma_n^{p+2} \end{vmatrix} \\ &= |\Gamma_n^p + \Gamma_n^{p+2}| \cdot \begin{vmatrix} -\Gamma_{n+1}^p & \left(\Gamma_{n+1}^{p+1} \right)^{(n+1)} \\ \left(\Gamma_{n+1}^{p+1} \right)_{(n+1)} & \Gamma_n^p \end{vmatrix} \\ &= |\Gamma_n^p + \Gamma_n^{p+2}| \cdot \begin{vmatrix} -\Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times n} \\ \left(\Gamma_{n+1}^{p+1} \right)_{(n+1)} & \Gamma_n^p + \Gamma_n^{p+2} \end{vmatrix}. \end{aligned}$$

Hence,

$$\left| \Delta_{3n+1}^{3p} \right| \equiv (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_n^p|^2.$$

12) Combine (2.4) and (2.7), we have

$$\begin{aligned} \left| P^t \Delta_{3n+2}^{3p} P \right| &\equiv \begin{vmatrix} -\Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} & \left(\Gamma_{n+1}^p \right)^{(n+1)} \\ \Gamma_{n+1}^{p+1} & \Gamma_{n+1}^p & -\left(\Gamma_{n+1}^{p+1} \right)^{(n+1)} \\ \left(\Gamma_{n+1}^p \right)_{(n+1)} & -\left(\Gamma_{n+1}^{p+1} \right)_{(n+1)} & \Gamma_n^{p+2} \end{vmatrix} \\ &= \begin{vmatrix} -\Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} & \mathbf{0}_{(n+1) \times n} \\ \Gamma_{n+1}^{p+1} & \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times n} \\ \left(\Gamma_{n+1}^p \right)_{(n+1)} & -\left(\Gamma_{n+1}^{p+1} \right)_{(n+1)} & \Gamma_n^p + \Gamma_n^{p+2} \end{vmatrix}. \end{aligned}$$

By Lemma 1,

$$\begin{aligned} \left| \Delta_{3n+2}^{3p} \right| &\equiv |\Delta_n^p| \cdot \begin{vmatrix} -\Gamma_{n+1}^p & \Gamma_{n+1}^{p+1} \\ \Gamma_{n+1}^{p+1} & \Gamma_{n+1}^p \end{vmatrix} \\ &= (-1)^n |\Gamma_{n+2}^p| \cdot |\Delta_n^p|^2 + (-1)^{n+1} |\Gamma_{n+1}^p| \cdot |\Delta_n^p| \cdot |\Delta_{n+1}^p|. \end{aligned}$$

13) Combine (2.4) and (2.5), we have

$$\begin{aligned}
\left| P^t \Delta_{3n}^{3p+1} P \right| &\equiv \begin{vmatrix} \Gamma_n^{p+1} & \Gamma_n^p & -\Gamma_n^{p+1} \\ \Gamma_n^p & -\Gamma_n^{p+1} & \Gamma_n^{p+2} \\ -\Gamma_n^{p+1} & \Gamma_n^{p+2} & \Gamma_n^{p+1} \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{0}_{n \times n} & \Gamma_n^p + \Gamma_n^{p+2} & \mathbf{0}_{n \times n} \\ \Gamma_n^p & -\Gamma_n^{p+1} & \Gamma_n^{p+2} \\ -\Gamma_n^{p+1} & \Gamma_n^{p+2} & \Gamma_n^{p+1} \end{vmatrix} \\
&= (-1)^n |\Gamma_n^p + \Gamma_n^{p+2}| \cdot \begin{vmatrix} \Gamma_n^p & \Gamma_n^{p+2} \\ -\Gamma_n^{p+1} & \Gamma_n^{p+1} \end{vmatrix} \\
&= (-1)^n |\Gamma_n^p + \Gamma_n^{p+2}| \cdot \begin{vmatrix} \Gamma_n^p + \Gamma_n^{p+2} & \Gamma_n^{p+2} \\ \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \end{vmatrix} \\
&= (-1)^n |\Gamma_n^p + \Gamma_n^{p+2}|^2 \cdot |\Gamma_n^{p+1}|.
\end{aligned}$$

Therefore,

$$|\Delta_{3n}^{3p+1}| \equiv (-1)^n |\Gamma_n^{p+1}| \cdot |\Delta_n^p|^2.$$

14) Combine (2.4) and (2.6), we have

$$\begin{aligned}
\left| P^t \Delta_{3n+1}^{3p+1} P \right| &\equiv \begin{vmatrix} \Gamma_{n+1}^{p+1} & (\Gamma_{n+1}^p)^{(n+1)} & -(\Gamma_{n+1}^{p+1})^{(n+1)} \\ (\Gamma_{n+1}^p)_{(n+1)} & -\Gamma_n^{p+1} & \Gamma_n^{p+2} \\ -(\Gamma_{n+1}^{p+1})_{(n+1)} & \Gamma_n^{p+2} & \Gamma_n^{p+1} \end{vmatrix} \\
&= \begin{vmatrix} \Gamma_{n+1}^{p+1} & (\Gamma_{n+1}^p)^{(n+1)} & \mathbf{0}_{(n+1) \times n} \\ (\Gamma_{n+1}^p)_{(n+1)} & -\Gamma_n^{p+1} & \Gamma_n^p + \Gamma_n^{p+2} \\ -(\Gamma_{n+1}^{p+1})_{(n+1)} & \Gamma_n^{p+2} & \mathbf{0}_{n \times n} \end{vmatrix} \\
&= (-1)^n |\Gamma_n^p + \Gamma_n^{p+2}| \cdot \begin{vmatrix} \Gamma_{n+1}^{p+1} & (\Gamma_{n+1}^p)^{(n+1)} \\ -(\Gamma_{n+1}^{p+1})_{(n+1)} & \Gamma_n^{p+2} \end{vmatrix} \\
&= (-1)^n |\Gamma_n^p + \Gamma_n^{p+2}| \cdot \begin{vmatrix} \Gamma_{n+1}^{p+1} & (\Gamma_{n+1}^p)^{(n+1)} \\ \mathbf{0}_{n \times (n+1)} & \Gamma_n^p + \Gamma_n^{p+2} \end{vmatrix}.
\end{aligned}$$

Hence,

$$|\Delta_{3n+1}^{3p+1}| \equiv (-1)^n |\Gamma_{n+1}^{p+1}| \cdot |\Delta_n^p|^2.$$

15) Combine (2.4) and (2.7), we have

$$\begin{aligned}
\left| P^t \Delta_{3n+2}^{3p+1} P \right| &\equiv \begin{vmatrix} \Gamma_{n+1}^{p+1} & \Gamma_{n+1}^p & -\left(\Gamma_{n+1}^{p+1}\right)^{(n+1)} \\ \Gamma_{n+1}^p & -\Gamma_{n+1}^{p+1} & \left(\Gamma_{n+1}^{p+2}\right)^{(n+1)} \\ -\left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} & \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \Gamma_n^{p+1} \end{vmatrix} \\
&= \begin{vmatrix} \Gamma_{n+1}^{p+1} & \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times n} \\ \Gamma_{n+1}^p & -\Gamma_{n+1}^{p+1} & \mathbf{0}_{(n+1) \times n} \\ -\left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} & \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \Gamma_n^{p+1} + \Gamma_n^{p+3} \end{vmatrix} \\
&= \left| \Gamma_n^{p+1} + \Gamma_n^{p+3} \right| \cdot \begin{vmatrix} \Gamma_{n+1}^{p+1} & \Gamma_{n+1}^p \\ \Gamma_{n+1}^p & -\Gamma_{n+1}^{p+1} \end{vmatrix}.
\end{aligned}$$

By Lemma 1,

$$\left| \Delta_{3n+2}^{3p+1} \right| \equiv (-1)^n \left| \Gamma_{n+2}^p \right| \cdot \left| \Delta_n^p \right| \cdot \left| \Delta_n^{p+1} \right| + (-1)^{n+1} \left| \Gamma_{n+1}^p \right| \cdot \left| \Delta_n^{p+1} \right| \cdot \left| \Delta_{n+1}^p \right|.$$

16) Combine (2.4) and (2.5), we have

$$\begin{aligned}
\left| P^t \Delta_{3n}^{3p+2} P \right| &\equiv \begin{vmatrix} \Gamma_n^p & -\Gamma_n^{p+1} & \Gamma_n^{p+2} \\ -\Gamma_n^{p+1} & \Gamma_n^{p+2} & \Gamma_n^{p+1} \\ \Gamma_n^{p+2} & \Gamma_n^{p+1} & -\Gamma_n^{p+2} \end{vmatrix} \\
&= \begin{vmatrix} \Gamma_n^p + \Gamma_n^{p+2} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ -\Gamma_n^{p+1} & \Gamma_n^{p+2} & \Gamma_n^{p+1} \\ \Gamma_n^{p+2} & \Gamma_n^{p+1} & -\Gamma_n^{p+2} \end{vmatrix} \\
&= \left| \Gamma_n^p + \Gamma_n^{p+2} \right| \cdot \begin{vmatrix} \Gamma_n^{p+2} & \Gamma_n^{p+1} \\ \Gamma_n^{p+1} & -\Gamma_n^{p+2} \end{vmatrix}.
\end{aligned}$$

By Lemma 1,

$$\left| \Delta_{3n}^{3p+2} \right| \equiv (-1)^n \left| \Gamma_n^{p+1} \right| \cdot \left| \Delta_n^p \right| \cdot \left| \Delta_n^{p+1} \right| + (-1)^{n+1} \left| \Gamma_{n+1}^{p+1} \right| \cdot \left| \Delta_n^p \right| \cdot \left| \Delta_{n-1}^{p+1} \right|.$$

17) Combine (2.4) and (2.6), we have

$$\begin{aligned}
\left| P^t \Delta_{3n+1}^{3p+2} P \right| &\equiv \begin{vmatrix} \Gamma_{n+1}^p & -\left(\Gamma_{n+1}^{p+1}\right)^{(n+1)} & \left(\Gamma_{n+1}^{p+2}\right)^{(n+1)} \\ -\left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} & \Gamma_n^{p+2} & \Gamma_n^{p+1} \\ \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \Gamma_n^{p+1} & -\Gamma_n^{p+2} \end{vmatrix} \\
&= \begin{vmatrix} \Gamma_{n+1}^p & \mathbf{0}_{(n+1) \times n} & \left(\Gamma_{n+1}^{p+2}\right)^{(n+1)} \\ -\left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} & \mathbf{0}_{n \times n} & \Gamma_n^{p+1} \\ \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \Gamma_n^{p+1} + \Gamma_n^{p+3} & -\Gamma_n^{p+2} \end{vmatrix} \\
&= (-1)^n \left| \Gamma_n^{p+1} + \Gamma_n^{p+3} \right| \cdot \begin{vmatrix} \Gamma_{n+1}^p & \left(\Gamma_{n+1}^{p+2}\right)^{(n+1)} \\ -\left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} & \Gamma_n^{p+1} \end{vmatrix} \\
&= (-1)^n \left| \Gamma_n^{p+1} + \Gamma_n^{p+3} \right| \cdot \begin{vmatrix} \Gamma_{n+1}^p & \left(\Gamma_{n+1}^{p+2}\right)^{(n+1)} \\ 0_{n \times (n+1)} & \Gamma_n^{p+1} + \Gamma_n^{p+3} \end{vmatrix}.
\end{aligned}$$

Hence,

$$|\Delta_{3n+1}^{3p+2}| \equiv (-1)^n |\Gamma_{n+1}^p| \cdot |\Delta_n^{p+1}|^2.$$

18) Combine (2.4) and (2.7), we have

$$\begin{aligned} |P^t \Delta_{3n+2}^{3p+2} P| &\equiv \begin{vmatrix} \Gamma_{n+1}^p & -\Gamma_{n+1}^{p+1} & \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)}^{(n+1)} \\ -\Gamma_{n+1}^{p+1} & \Gamma_{n+1}^{p+2} & \left(\Gamma_{n+1}^{p+1}\right)_{(n+1)}^{(n+1)} \\ \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} & -\Gamma_n^{p+2} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & -\Gamma_{n+1}^{p+1} & \mathbf{0}_{(n+1) \times n} \\ -\Gamma_{n+1}^{p+1} & \Gamma_{n+1}^{p+2} & \left(\Gamma_{n+1}^{p+1} + \Gamma_{n+1}^{p+3}\right)_{(n+1)}^{(n+1)} \\ \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} & \mathbf{0}_{n \times n} \end{vmatrix} \\ &= \begin{vmatrix} \Gamma_{n+1}^p & -\Gamma_{n+1}^{p+1} & \mathbf{0}_{(n+1) \times n} \\ \mathbf{0}_{n \times (n+1)} & \mathbf{0}_{n \times (n+1)} & \left(\Gamma_{n+1}^{p+1} + \Gamma_{n+1}^{p+3}\right)_{(n+1)}^{(n+1)} \\ -\left(\alpha_{p+n+1}^{n+1}\right)^t & \left(\alpha_{p+n+2}^{n+1}\right)^t & \mathbf{0}_{n \times n} \end{vmatrix} \\ &= |\Gamma_n^{p+1} + \Gamma_n^{p+3}| \cdot \begin{vmatrix} \Gamma_{n+1}^p & -\Gamma_{n+1}^{p+1} \\ -\left(\alpha_{p+n+1}^{n+1}\right)^t & \left(\alpha_{p+n+2}^{n+1}\right)^t \end{vmatrix} \\ &= |\Gamma_n^{p+1} + \Gamma_n^{p+3}| \cdot (-1) \begin{vmatrix} \left(\Gamma_{n+2}^p\right)_{(n+1)}^{(n+2)} & -\left(\Gamma_{n+2}^{p+1}\right)_{(n+1)}^{(n+2)} \\ \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} \end{vmatrix}. \end{aligned}$$

Note that

$$\begin{aligned} &\begin{vmatrix} \left(\Gamma_{n+2}^p\right)_{(n+1)}^{(n+2)} & -\left(\Gamma_{n+2}^{p+1}\right)_{(n+1)}^{(n+2)} \\ \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \left(\Gamma_{n+1}^{p+1}\right)_{(n+1)} \end{vmatrix} \\ &= \begin{vmatrix} \left(\Gamma_{n+2}^p\right)_{(n+1)}^{(n+2)} & \mathbf{0}_{(n+2) \times n} & -\alpha_{p+n+1}^{n+2} \\ \left(\Gamma_{n+1}^{p+2}\right)_{(n+1)} & \Gamma_n^{p+1} + \Gamma_n^{p+3} & \alpha_{p+n+1}^n \end{vmatrix} \\ &= (-1)^n |\Gamma_n^{p+1} + \Gamma_n^{p+3}| \cdot (-1) |\Gamma_{n+2}^p|, \end{aligned}$$

we have

$$|\Delta_{3n+2}^{3p+2}| \equiv (-1)^n |\Gamma_{n+2}^p| \cdot |\Delta_n^{p+1}|^2.$$

□

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